CHAPTER - 13

LIMITS AND DERIVATIES

KEY POINTS

- To check whether limit of f(x) as x approaches to exists i.e., $\lim_{x\to c} f(x)$ exists, we proceed as follows.

  (i) Find L.H.L at x = a using L.H.L. = $\lim_{h\to 0} f(a-h)$.

  (ii) Find R.H.L at x = a using R.H.L. = $\lim_{h\to 0} f(a+h)$.

  (iii) If both L.H.L. and R.H.L. are finite and equal, then limit at x = a i.e., $\lim_{x\to a} f(x)$ exists and equals to the value obtained from L.H.L or R.H.L else we say "limit does not exist".

- $\lim_{x\to c} f(x) = l$, if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = l$

- $\lim_{x\to c} a = a$, where a is a fixed real number.

- $\lim_{x\to c} x^n = c^n$, for all $n \varepsilon N$

▶ **ALGEBRA OF LIMITS:** Let f, g be two functions such that $\lim_{x\to c} f(x) = l$, and $\lim_{x\to c} g(x) = m$.

- $\lim_{x\to c}[\alpha f(x)] = \alpha \lim_{x\to c} f(x) = \alpha l$, for all $\alpha \varepsilon R$

- $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = l \pm m$

- $\lim_{x\to c}[f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = l \cdot m$

- $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{l}{m}, \ m \neq 0, \ g(x) \neq 0$

- $\lim_{x \to c} \frac{1}{f(x)} = \frac{1}{\lim_{x \to c} f(x)} = \frac{1}{l}, \ l \neq 0, f(x) \neq 0$

- $\lim_{x \to c} [f(x)]^n = [(\lim_{x \to c} f(x)]^n = l^n$, for all $n \ \varepsilon \ N$

▶ SOME IMPORTANT RESULTS ON LIMITS:

- $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$

- $\lim_{x \to 0} \frac{\tan x}{x} = 1$

- $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

- $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

- $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

- $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$

- $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(-x)$

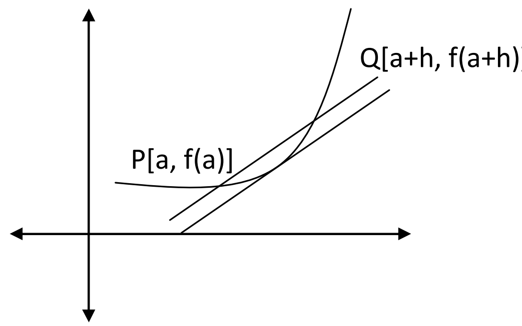
SOME IMPORTANT RESULTS ON DERIVATIVE:

- $\dfrac{d(\sin x)}{dx} = \cos x$
- $\dfrac{d(\cos x)}{dx} = -\sin x$
- $\dfrac{d(\tan x)}{dx} = \sec^2 x$
- $\dfrac{d(\cot x)}{dx} = -\text{cosec}^2 x$
- $\dfrac{d(\sec x)}{dx} = \sec x \cdot \tan x$
- $\dfrac{d(\cos ecx)}{dx} = -\text{cosec}x \cdot \cot x$
- $\dfrac{d(x^n)}{dx} = n \cdot x^{n-1}$
- $\dfrac{d(\sqrt{x})}{dx} = \dfrac{1}{2\sqrt{x}}$
- $\dfrac{d(a)}{dx} = 0,\ a = \text{constan } t$
- $\dfrac{d(e^x)}{dx} = e^x$
- $\dfrac{d(\log x)}{dx} = \dfrac{1}{x}$
- $\dfrac{d(a^x)}{dx} = a^x \cdot \log a$

Logarithm Properties:

- $\log_e(A \cdot B) = \log_e A + \log_e B$
- $\log_e\left(\dfrac{A}{B}\right) = \log_e A - \log_e B$
- $\log_e(A^m) = m \cdot \log_e A$
- $\log_a(1) = 0$
- $\log_B(A) = x$, then $B^x = A$

- Let y = f(x) be a function defined in some neighbourhood of the point 'a'. Let P[a, f(a)] and Q[a + h, f(a + h)] are two points on the graph of f(x) where h is very small and h ≠ 0.



$$\text{Slope of } PQ = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- If $\lim_{h \to 0}$ point Q approaches to P and the line PQ becomes a tangent to the curve at point P.

  $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ (if exists) is called derivative of f(x) at the point 'a'.

  It is denoted by f'(a).

▶ ALGEBRA OF DERIVATIVES:

- $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$, where c is a constan t

- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

Product Rule:

- $\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$

Quotient Rule:

- $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\cdot\frac{d}{dx}[f(x)]-f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

- If y = f(x) is a given curve then slope of the tangent to the curve at the point (h,k) is given by $\frac{dy}{dx}|_{(h,k)}$ and is denoted by 'm'

SECTION - A

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. Fill in the blanks in each of the followings:

(a) $\lim_{x\to 2^-}[x]=$ ________________.

(b) $\lim_{x\to 2^+}[x]=$ ________________.

(c) $\lim_{x\to 2^-}\frac{|x|}{x}=$ ________________.

(d) $\lim_{x\to 2^+}\frac{|x|}{x}=$ ________________.

(e) If f(x) = $\text{Sin}^2 x$, then derivative of f(x) is ____________.

2. State whether the following statements are True or False.

(a) If $L=\lim_{x\to 0}\frac{\sin 2x}{\tan 3x}$; So the value of L is 1.

(b) If $L=\lim_{x\to 0}\frac{\sin x-\text{tann}\,x}{x^3}$; So the value of 2L is 1.

(c) If f(x) = $x^2 - 3x + 1$, then derivative of f(x) at x = 2, f '(2) = 1.

(d) $\lim_{x \to 3} [x]$ exists and equal to 3.

(e) $\lim_{x \to 3} |x|$ exists and equal to 3

Note: Q.3 – Q.10 are Multiple Choice Questions (MCQ), select the correct alternatives out of given four alternatives in each.

3. $\lim_{x \to \pi} \frac{\sin x}{x - \pi}$ is -

(a) 1 (b) 2

(c) –1 (d) does not exist.

4. If $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80,$ then n is -

(a) 2 (b) 3

(c) 4 (d) 5.

5. If $L = \lim_{x \to 1} \frac{x^4 - 1}{x^3 - 1},$ then 3L is -

(a) 2 (b) 3

(c) 4 (d) None of these.

6. $\lim_{x \to 0} \frac{(1+x)^{16} - 1}{(1+x)^4 - 1}$ is -

(a) 0 (b) 4

(c) 8 (d) 16.

7. $\lim_{x \to 1} \frac{x + x^2 + x^3 + x^4 - 4}{x - 1}$ is -

(a) 0 (b) 4

(c) 10 (d) Does not exist.

8. $\lim_{x \to \frac{\pi}{4}} \frac{\sec^2 x - 2}{\tan x - 1}$ is -

(a) 0 (b) 1

(c) 2 (d) 4.

9. If $y = \sin^4 x + \cos^4 x$, then $\frac{dy}{dx} =$

(a) $4\sin^3 x + 4\cos^3 x$ (b) $4\sin^3 x - 4\cos^3 x$

(c) $-\sin 4x$ (d) 0.

10. Match the following:

	Column-1		Column-2
A	$\lim_{x \to \infty} \frac{1+2+3+\ldots+x}{x^2} =$	P	$\frac{1}{3}$
B	$\lim_{x \to \infty} \frac{1+4+9+\ldots+x^2}{x^3} =$	Q	$\frac{1}{3}$
C	$\lim_{x \to \infty} \frac{1+8+27+\ldots+x^3}{x^4} =$	R	$\frac{1}{2}$

Which one of the following is true?

(a) A → P, B→ Q, C→ R

(b) A → Q, B→ P, C→ R

(c) A → Q, B→ R, C→ P

(d) A → R, B→ P, C→ Q

11. Evaluate $\lim_{x \to 0} \frac{(1+x)^m - 1}{(1+x)^n - 1}$

12. Evaluate $\lim_{x \to 0} \frac{(\sin 2x) + 3x}{2x + (\tan 3x)}$

13. Evaluate $\lim_{x \to 0} \frac{1 - \cos 2x}{1 - \cos 4x}$

14. If y = $\sin^2 x.\cos^3 x$, then $\frac{dy}{dx}$.

15. If y = sin 2x.cos 3x, then $\frac{dy}{dx}$.

SECTION - B

SHORT ANSWER TYPE QUESTIONS (2 MARKS)

16. Differentiate $\frac{\sin x}{x}$ with respect to x.

17. Differentiate $x^3 + 3^3 + 3^x$ with respect to x.

18. Differentiate $\sin^2(x^3 + x - 1) + \frac{1}{\sec^2(x^3 + x - 1)}$ with respect to x.

19. Differentiate $\left(\frac{x^a}{x^b}\right)^{a+b} \cdot \left(\frac{x^b}{x^c}\right)^{b+c} \cdot \left(\frac{x^c}{x^a}\right)^{c+a}$ with respect to x.

20. Differentiate $\frac{1}{1 + x^{b-a} + x^{c-a}} + \frac{1}{1 + x^{a-b} + x^{c-b}} + \frac{1}{1 + x^{a-c} + x^{b-c}}$ w.r.t to x.

21. Find the derivative of x using first principle method.

22. If $\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$, then find the value of k.

23. Evaluate: $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$.

24. Find the derivative of (x – 1)(x + 1)(x^2 + 1)(x^4 + 1) with respect to x.

25. Differentiate $\frac{x^8 - 1}{x^4 - 1}$ with respect to x.

SECTION – C

LONG ANSWER TYPE - I QUESTIONS (4 MARKS)

26. Differentiate Sin^2 x with respect to x using First principle method.

27. Differentiate Sin (x^2) with respect to x using First principle method.

Differentiate the following with respect to x using First principle method. (For Q. 28 – 35)

28. $\cos\sqrt{x}$

29. $\sqrt{\tan x}$

30. $\sec^3 x$

31. $\operatorname{cosec}(2x+3)$

32. $\sin^{\frac{1}{3}} x = \sqrt[3]{\sin x}$

33. $\dfrac{x^2}{x+1}$

34. $\dfrac{2x+3}{x+1}$

35. $\sqrt{x} + \dfrac{1}{\sqrt{x}}$

Evaluate the following Limits: (For Q. 36 – 53)

36. $\lim\limits_{x \to \infty} \dfrac{2x^8 - 3x^2 + 1}{x^8 + 6x^5 - 7}$

37. $\lim\limits_{x \to 1} \dfrac{2x^8 - 3x^2 + 1}{x^8 + 6x^5 - 7}$

38. $\lim_{x\to 0} \frac{1-\cos 2x}{x \cdot \tan 3x}$

39. $\lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$

40. $\lim_{x\to \frac{\pi}{6}} \frac{\sqrt{3}\sin x - \cos x}{\frac{\pi}{6} - x}$

41. $\lim_{x\to 0} \frac{\sin x}{\tan x^0}$ (where x^0 represents x degree)

42. $\lim_{x\to 9} \frac{x^{\frac{3}{2}} - 27}{x^2 - 81}$

43. $\lim_{x\to a} \frac{(x+2)^{\frac{5}{2}} - (a+2)^{\frac{5}{2}}}{x-a}$

44. $\lim_{x\to 0} \frac{\cos ax - \cos bx}{1-\cos x}$

45. $\lim_{x\to a} \frac{\cos x - \cos a}{\cot x - \cot a}$

46. $\lim_{x\to \pi} \frac{1+\sec^3 x}{\tan^2 x}$

47. $\lim_{x\to 1} \frac{x-1}{\log_e x}$

48. $\lim_{x\to e} \frac{x-e}{(\log_e x)-1}$

49. $\lim_{x \to 2} \left[\frac{4}{x^3 - 2x^2} + \frac{1}{2 - x} \right]$

50. $\lim_{x \to a} \left[\frac{\sqrt{a + 2x} - \sqrt{3x}}{\sqrt{3a + x} - 2\sqrt{x}} \right]$

51. $\lim_{x \to 0} \frac{\sin(2 + x) - \sin(2 - x)}{x}$

52. $\lim_{x \to 0} \frac{1 - \cos x \cdot \sqrt{\cos 2x}}{\sin^2 x}$

53. $\lim_{x \to 0} \frac{6^x - 2^x - 3^x + 1}{\log(1 + x^2)}$

54. Differentiate the following w.r.t.

(a) $\frac{(x-1)(x-2)(x-3)}{x^2 - 5x + 6}$

(b) $\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$

(c) $\frac{x \sin x + \cos x}{x \sin x - \cos x}$

(d) $x \cdot \sin x \cdot e^x$

55. Find the values of a and b if $\lim_{x \to 2} f(x)$ and $\lim_{x \to 4} f(x)$ exists where

$$f(x) = \begin{vmatrix} x^2 + ax + b, & 0 \le x < 2 \\ 3x + 2, & 2 \le x \le 4 \\ 2ax + 5b, & 4 < x < 8 \end{vmatrix}$$

ANSWERS

1. (a) 1
   (b) 2
   (c) –1
   (d) 1
   (e) sin2x

2. (a) False
   (b) False
   (c) True
   (d) False
   (e) True

3. (c)

4. (d)

5. (c)

6. (b)

7. (c)

8. (c)

9. (c)

10. (d)

11. $\frac{m}{n}$

12. 1

13. $\frac{1}{4}$

14. $\cos^2 x.\sin x(2\cos^2 x - 3\sin^2 x)$

15. $2\cos 2x.\cos 3x - 3\sin 2x.\sin 3x$

16. $\frac{x\cos x - \sin x}{x^2}$

17. $3x^2 + 3x.\log 3$

18. 0

19. 0

20. 0

21. 1

22. $\frac{8}{3}$

23. $\frac{1}{2}$

24. $8x^7$

25. $4x^3$

26. sin2x

27. $2x.\cos(x)^2$

28. $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

29. $\frac{\sec^2 x}{2\sqrt{\tan x}}$

30. $3\sec^3 x.\tan x$

31. $-2\text{cosec}(2x + 3).\cot(2x + 3)$

32. $\frac{\cos x}{3\sqrt[3]{\sin^2 x}}$

33. $\frac{x^2 + 2x}{(x+1)^2}$

34. $\frac{-1}{(x+1)^2}$

35. $\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$

36. 2

37. $\frac{5}{17}$

38. $\frac{2}{3}$

39. $\sqrt{2}$

40. 2

41. $\frac{180^0}{\pi}$

42. $\frac{1}{4}$

43. $\frac{5(a+2)^{\frac{3}{2}}}{2}$

44. $b^2 - a^2$

45. $\sin^3 a$

46. $\frac{-3}{2}$

47. 1

48. e

49. –1

50. $\frac{1}{\sqrt{3}}$

51. 2cos2

52. $\frac{3}{2}$

53. (log2)(log3)

54. (a) 1

(b) $8x^7 + 8x^{-9}$

(c) $\frac{-2(x + \sin x \cdot \cos x)}{(x\sin x - \cos x)^2}$

(d) $e^x (x\sin x + x\cos x + \sin x)$

55. a = –1, b = 6