CHAPTER - 5

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

KEY POINTS

- The imaginary number $\sqrt{-1} = i$, is called iota
- For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$
- $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$ if both a and b are negative real numbers
- A number of the form z = a + ib, where a, b $\in$ R is called a complex number.

  a is called the real part of z, denoted by Re(z) and b is called the imaginary part of z, denoted by Im(z)
- a + ib = c + id if a = c, and b = d
- $z_1 = a + ib$, $z_2 = c + id$.

  In general, we cannot compare and say that $z_1 > z_2$ or $z_1 < z_2$ but if b, d = 0 and a > c then $z_1 > z_2$

  i.e. we can compare two complex numbers only if they are purely real.
- 0 + *i* 0 is additive identity of a complex number.
- –z = –a –ib is called the Additive Inverse or negative of z = a + ib
- 1 + i 0 is multiplicative identity of complex number.

- $\bar{z} = a - ib$ is called the conjugate of z = a + ib
- $i^0 = 1$
- $z^{-1} = \frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$ is called the multiplicative Inverse of z = a + ib (a ≠ 0, b ≠ 0)
- The coordinate plane that represents the complex numbers is called the complex plane or the Argand plane
- Polar form of z = a + ib is,

  z = r (cosθ + i sinθ) where $r = \sqrt{a^2 + b^2} = |z|$ is called the modulus of z,θ is called the argument or amplitude of z.
- The value of θ such that, –π < θ < π is called the principle argument of z.
- Z = x + iy, x > 0 and y > 0 the argument of z is acute angle given by tanα $= \frac{y}{x}$

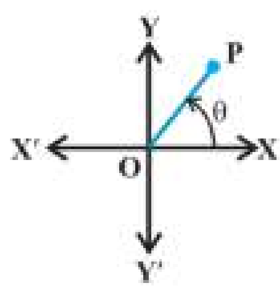
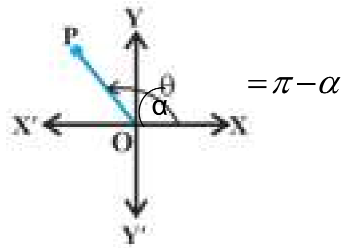


figure (i)

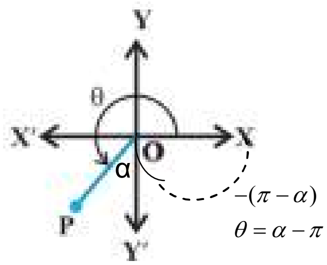
- $Z = x + iy$, $x < 0$ and $y > 0$ the argument of z is $\pi - \alpha$, where α is acute angle given by $\tan\alpha = \left|\frac{y}{x}\right|$



$\theta = \pi - \alpha$

figure (ii)

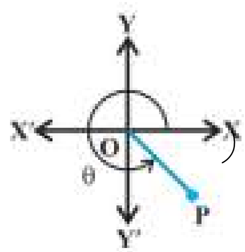
- $Z = x + iy$, $x < 0$ and $y < 0$ the argument of z is $\alpha - \pi$, where π is acute angle given by $\tan\alpha = \left|\frac{y}{x}\right|$



$\theta = \alpha - \pi$

figure (iii)

- $Z = x + iy$, $x > 0$ and $y < 0$ the argument of z is $-\alpha$, where α is acute angle given by $\tan\alpha = \left|\frac{y}{x}\right|$



$\theta = -\alpha$

figure (iv)

- $|z_1 + z_2| \le |z_1| + |z_2|$
- $|z_1 z_2| = |z_1| . |z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$; $|z^n| = |z|^n$; $|z| = |\bar{z}| = |-z| = |-\bar{z}|$; $z\bar{z} = |z|^2$
- $|z_1 - z_2| \ge |z_1| - |z_2|$
- If $z_1 = r_1 (\cos \theta_1 + i\sin \theta_1)$

  $z_2 = r_2 (\cos \theta_2 + i\sin \theta_2)$

  then $z_1 z_2 = r_1 r_2[\cos (\theta_1 + \theta_2) + i\sin (\theta_1 + \theta_2)]$

  $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$
- For the quadratic equation $ax^2 + bx + c = 0$,

  $a, b, c \in R$, $a \ne 0$, if $b^2 - 4ac < 0$

  then it will have complex roots given by,

  $$x = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

W. R. Hamilton
(1805-1865)

- $\sqrt{a+ib}$ is called square root of $z = a + ib$, $\therefore \sqrt{a+ib} = x + iy$

  squaring both sides we get $a + ib = x^2 - y^2 + 2i(xy)$

  $x^2 - y^2 = a$, $2xy = b$. Solving these we get x and y.

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. Write the value of $i + i^{10} + i^{20} + i^{30}$

2. Write the additive Inverse of $6i - i\sqrt{-49}$

3. Write the multiplicative Inverse of $1 + 4\sqrt{3}\ i$

4. Write the conjugate of $\dfrac{2-i}{(1-2i)^2}$

5. Write the amplitude of $\dfrac{1}{i}$

6. Write the Argument of $(1+\sqrt{3}i)(\cos\theta + i\sin\theta)$

7. Write in the form of $a + ib$: $\dfrac{1}{-2+\sqrt{-3}}$

8. Write the argument of $-i$

9. Write the value of arg(z) + arg$(\bar{z})$

10. Multiply by its $2 - 3\,i$ conjugate.

11. If $\sqrt{7-24i} = x + iy$ and $x = \pm 4,\ \ y = \pm 3$ then $\sqrt{7-24i}$ = ?

12. What is the least integral value of K which makes the roots of the equation $x^2 + 5x + k = 0$ imaginary?

Fill in the blanks (Exercise 13 to 17) :-

13. The real value of 'a' for which $3i^3 - 2ai^2 + (1-a)i$ is real is ____.

14. If |z| = 2 and arg(z) = $\dfrac{\pi}{4}$, then z = ______.

15. The value of $\left(-\sqrt{-1}\right)^{4n-3}$, when n $\in$ N, is _____.

16. If a complex number lies in the third quadrant, then its conjugate lies in the _______ quadrant.

17. The value of $\sqrt{-25} \times \sqrt{-9}$ is ______.

State true or false for the following statements (Exercise 18 to 22) :-

18. The order relation is defined on the set of complex number

19. Multiplication of a non-zero complex number by $-i$ rotates the point about origin through a right angle n anti-clock wise direction.

20. z is not a complex number.

21. The complex number $\cos\theta + i\sin\theta$ can be zero for some 'θ'.

22. The argument of the complex number $z = (1 + i\sqrt{3})(1 + i)$ $(\cos\theta + i\sin\theta)$ is $\frac{7\pi}{12} + \theta$.

23. Match the following statements of column A and B

A	B
(a) The polar form of $i + \sqrt{3}$ is	(i) Purely real complex number
(b) The amplitude of $-1 + \sqrt{-3}$ is	(ii) Forth quadrant
(c) Reciprocal of $1 - i$ lies in	(iii) First quadrant
(d) Conjugate of $1 + i$ lies in	(iv) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
(e) The value of $1 + i^2 + i^4 + i^6 + \ldots + i^{20}$ is	(v) $\frac{2\pi}{3}$

SHORT ANSWER TYPE QUESTIONS (2 MARKS)

24. Evaluate :

(i) $\sqrt{-16}+3\sqrt{-25}+\sqrt{-36}-\sqrt{-625}$

(ii) $i\sqrt{-16}+i\sqrt{-25}+\sqrt{49}-i\sqrt{-49}+14$

(iii) $\left(i^{77}+i^{70}+i^{87}+i^{414}\right)^3$

(iv) $\dfrac{(3+\sqrt{5}i)(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$

25. Find x and y if (x + iy) (2 – 3i) = 4 + i.

26. If n is any positive integer, write value of $\dfrac{i^{4n+1}-i^{4n-1}}{2}$

27. If $z_1 = \sqrt{2}(\cos 30° + i \sin 30°)$, $z_2 = \sqrt{3}(\cos 60 + i \sin 30°)$

Find $R_e\,(z_1z_2)$

28. If $|z+4| \le 3$ then find the greatest and least values of $|z+1|$.

29. Find the real value of a for which $3i^3-2ai^2+(1-a)i+5$ is real.

30. If $\arg(z-1)=\arg(z+3i)$ where $z=x+iy$ find $x-1:y$.

31. If z = x + iy and the amplitude of $(z-2-3i)$ is $\dfrac{\pi}{4}$. Find the relation between x and y.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

32. If x + iy = $\sqrt{\dfrac{1+i}{1-i}}$ prove that $x^2 + y^2 = 1$

33. Find real value of θ such that, $\dfrac{1+i\cos\theta}{1-2i\cos\theta}$ is a real number.

34. If $\left|\frac{z-5i}{z+5i}\right|=1$ show that z is a real number.

35. If $x_n=\cos\frac{\pi}{2^n}+i\sin\frac{\pi}{2^n}$ Prove that $x_1\,x_2.......x_\infty=-1$

36. Find real value of x and y if $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$.

37. If $(1+i)(1+2i)(1+3i).......(1+ni)=x+iy$.

    Show, $2.5.10..........(1+n^2)=x^2+y^2$

38. If z = 2 – 3i show that $z^2 - 4z + 13 = 0$, hence find the value of $4z^3 - 3z^2 + 169$.

39. If $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=a+ib$, find a and b.

40. For complex numbers $z_1 = 6 + 3i$, $z_2 = 3 - i$ find $\frac{z_1}{z_2}$.

41. If $\left(\frac{2+2i}{2-2i}\right)^n=1$, find the least positive integral value of n

42. If $(x+iy)^{\frac{1}{3}}=a+ib$ prove $\left(\frac{x}{a}+\frac{y}{b}\right)=4\left(a^2-b^2\right)$.

43. Convert the following in polar form:

    (i) $-3\sqrt{2}+3\sqrt{2}i$ (ii) $\frac{\left(\sqrt{3}-1\right)-\left(\sqrt{3}+1\right)i}{2\sqrt{2}}$

    (iii) $i(1+i)$ (iv) $\frac{5-i}{2-3i}$

44. Solve

    (i) $x^2-\left(3\sqrt{2}-2i\right)x-6\sqrt{2}i=0$ (ii) $x^2-(7-i)x+(18-i)=0$

45. Find the square root of $7-30\sqrt{-2}$.

46. Prove that $x^2 + 4 = (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$.

47. Show that $\left|\frac{z-2}{z-3}\right| = 2$ represent a circle find its centre and radius.

48. Find all non-zero complex number z satisfying $\bar{z} = iz^2$.

49. If $iz^3 + z^2 - z + i = 0$ then show that $| z | = 1$.

50. If z_1, z_2 are complex numbers such that, $\frac{2z_1}{3z_2}$ is purely imaginary number then find $\left|\frac{z_1 - z_2}{z_1 + z_2}\right|$.

51. If z_1 and z_2 are complex numbers such that,

$\left|1-\bar{z}_1 z_2\right|^2 - \left|z_1 - z_{2^2}\right|^2 = k\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$. Find value of k.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

52. Find number of solutions of $z^2 + |z|^2 = 0$.

53. If z_1, z_2 are complex numbers such that $\left|\frac{z_1 - 3z_2}{3 - z_1.\bar{z}_2}\right| = 1$ and $|z_2| \neq 1$ then find $|z_1|$.

54. Evaluate $x^4 - 4x^3 + 4x^2 + 8x + 44$, When $x = 3 + 2i$

55. If z_1, z_2 are complex numbers, both satisfy $z + \bar{z} = 2|z-1|$

$\arg|z_1 - z_2| = \frac{\pi}{4}$, then find $\operatorname{Im}(z_1 + z_2)$.

56. Solve $2x^2 - (3 + 7i)x - (3 - 9i) = 0$

57. What is the locus of z if amplitude of $z - 2 - 3i$ is $\frac{\pi}{4}$.

58. If $z = x + iy$ and $w = \frac{1 - iz}{z - i}$ show that if $|w| = 1$ then z is purely real.

59. Express the complex number in the form $r(\cos\theta + i\sin\theta)$

(i) $1 + i\tan\alpha$

(ii) $1 - \sin\alpha + i\cos\alpha$

60. If $\left(\frac{1+i}{1+2^2 i}\right)\times\left(\frac{1+3^2 i}{1+4^2 i}\right)\times........\times\left(\frac{1+(2n-1)^2 i}{1+(2n)^2 i}\right) = \frac{a+ib}{c+id}$ then show that $\frac{2}{17}\times\frac{82}{257}\times.......\times\frac{1+(2n-1)^4}{1+(2n)^4} = \frac{a^2+b^2}{c^2+d^2}$.

61. Find the values of x and y for which complex numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate to each other.

62. The complex number z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a equilateral triangle.

63. If $f(z) = \frac{7 - z}{1 - z^2}$ where $z = 1 + 2i$ then show that $|f(z)| = \frac{|z|}{2}$.

64. If z_1, z_2, z_3 are complex numbers such that

$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$ then find the value of $|z_1 + z_2 + z_3|$

ANSWERS

1. $-1 + i$
2. $-7 - 6i$
3. $\frac{1}{49} - \frac{4\sqrt{3}i}{49}$
4. $\frac{-2}{25} + \frac{11i}{25}$
5. $\frac{-\pi}{2}$
6. $\theta + \frac{\pi}{3}$
7. $\frac{-2}{7} - \frac{i\sqrt{3}}{7}$
8. $\frac{-\pi}{2}$
9. 0
10. 13
11. $-4 + 3i$ and $4 + 3i$
12. 7
13. –2
14. $z = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
15. $-i$
16. First
17. –15
18. False
19. False
20. False
21. False
22. True
23. (a) → (iv)
    (b) → (v)
    (c) → (ii)
    (d) → (iii)
    (e) → (i)
24. (i) 0
    (ii) 19
    (iii) –8
    (iv) $\frac{-7}{\sqrt{2}}i$
25. $x = \frac{5}{13}, \quad y = \frac{14}{13}$
26. i
27. 0 (zero)
28. 6 and zero
29. a = –2
30. 1 : 3

31. Locus of z is straight line i.e., x – y + 1 = 0

33. $\theta = (2n+1)\frac{\pi}{2}$

36. x = 3, y = –1

38. zero

39. a = 0, b = –2

40. $\frac{z_1}{z_2} = \frac{3(1+i)}{2}$

41. n = 4

43. (i) $6\left(\cos\frac{3\pi}{4} + i\sin\frac{\pi}{4}\right)$

(ii) $1\left[\cos\left(\frac{-5\pi}{12}\right) + i\sin\left(\frac{-5\pi}{12}\right)\right]$

(iii) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(iv) $\sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

44. (i) $3\sqrt{2}$ and $-2i$

(ii) $4-3i$ and $3+2i$

45. $\pm\left(5-3\sqrt{2}\,i\right)$

47. Centre $\left(\frac{10}{3}, 0\right)$ and radius = $\frac{2}{3}$

48. $z = 0,\ i,\ \frac{\sqrt{3}}{2} - \frac{1}{2}i,\ -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

50. 1

51. K = 1

52. Infinitely many solutions of the form $z = 0 \pm iy;\ y \in R$

53. $|z_1| = \sqrt{x^2 + y^2}$

54. 5

55. 2

56. $\frac{3}{2} + \frac{1}{2}i$ and $3i$

57. x – y + 1 = 0 straight line

59. (i) $\sec\alpha(\cos\alpha + i\sin\alpha)\ ,\ 0 \le \alpha < \frac{\pi}{2}$

$-\sec\alpha[\cos(\alpha - \pi) + i\sin(\alpha - \pi)],\ \frac{\pi}{2} < \alpha \le \pi$

(ii) $\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left[\cos\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right]$ if $0 \le \alpha < \frac{\pi}{2}$

$-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left[\cos\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)\right]$ if $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$

$-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left[\cos\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)\right]$ if $\frac{3\pi}{2} < \alpha < 2\pi$

60. When x = 1, y = –4 or x = –1, y = –4

61. 1 (one)